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RESEARCHES REGARDING MATHEMATICAL MODELING OF THE FRONTAL COLLISION OF TWO VEHICLES

Virgil POPA[1], Horia BELES[2], Bogdan TOLEA[3]
[1]Asirom VIG Alba Iulia, popa.virgil@gmail.com
[2]University of Oradea, horia.beles@gmail.com
[3]Transilvania University of Brasov, bogdan.tzolea@gmail.com

***Abstract*—**In this paper we present an integrated mathematical model for the study of frontal collision between two vehicles, incorporating the impact of the two vehicles and the impact between the occupant and the vehicle. Such mathematical model developed determines occupant kinematic parameters, ie acceleration, velocity and movement of the thorax and pelvis, depending on: the primary impact parameters vehicle – vehicle, physical characteristics and occupant belt stiffness.

***Keywords*—**frontal collision, collision force, kinetic energy, vehicle deformations, integrated mathematical model

I. Introduction

A significant contribution regarding to improve road safety is the integration of active and passive safety systems into a single system. In principle, this is achieved through the development of control algorithms, where the parameters as the force values that act upon the occupant during the impact, or activation of time adaptive restraint systems, are defined based on the response of the vehicle in collision. The aim is that the reference values of these forces to minimize the occupant acceleration [1].

Pulse of collision recorded in the interior reflects the real conditions of the impact according to specific parameters as:

1) *type of collision, as the main force of impact and type of the object collides with vehicle (vehicle or other obstacle fixed or mobile);*
2) *compatibility "partners" of the collision in terms of geometric and structural rigidity of the masses;*
3) *the relative velocity of collision.*

Considering the high percentage of frontal collision between two vehicles and high occupant injury risk in these collisions, mathematical modeling objective is to provide a mathematical model incorporating frontal impact vehicle-vehicle and vehicle-occupant impact.

Mathematical modeling developed in this paper aims to determine the parameters of the vehicle and occupant kinematics that can be used to assess the risk of occupant injury in the event of a frontal impact.

Designing mathematical model was made in three stages [2], [3]:

1) *mathematical modeling of the frontal impact “vehicle-vehicle” type;*
2) *mathematical modeling of the impact of occupant-vehicle frontal impact of the vehicle;*
3) *achieving integrated model incorporating both sub-models.*

II. Vehicle-Ocupant System Analysis. Kinetic Energy of the Occupant

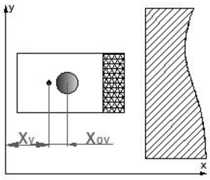
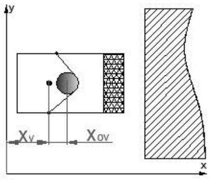
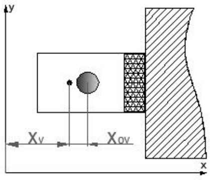
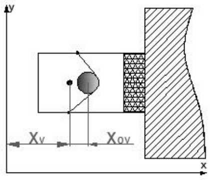
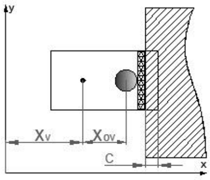
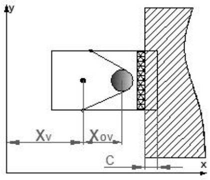
Considering vehicle non-deformable barrier collision to describe the kinetic energy of the vehicle and the occupant relative kinetic energy, using a vehicle-occupant model simplified. In this model, the occupant is represented by a mass point of mass m, and the vehicle is represented by a body of mass M, with deformable front. Movement inside the vehicle occupant may be free or restricted restraint systems (seat belts and airbags).

In Table I are shown the successive moments of vehicle-barrier impact and occupant-vehicle as well as calculation of kinetic energy relationships and kinetic energy of the vehicle relative to the vehicle occupant. Are two distinct situations: when the occupant moves freely in the car, or if movement is restricted by the occupant restraint systems. Since the barrier is fixed and non-deformable, from the moment of the first contact between the vehicle and barrier, the movement of the vehicle is equal to dynamic deformation C.

Significance of notations involved in relationships in Table I is as follows:

M – mass of the vehicle;
m – mass of the occupant;
x_v, x_o – movement of the vehicle respectively occupant movement;
C – dynamic deformation;
x_{ov} – relative movement vehicle-occupant;
d – space of the occupant on the direction of impact.

TABLE I
SUCCESIVE STEPS OF THE IMPACT VEHICLE-BARRIER AND OCCUPANT-VEHICLE

System status of vehicle-occupant		Kinetic energy of the vehicle: $E_{Cv} = \frac{1}{2} \cdot (M + m) \cdot \dot{x}_v^2$ Relative kinetic energy of the occupant: $E_{Cov} = \frac{1}{2} \cdot m \cdot \dot{x}_{ov}^2$	
Unbelted occupant	Belted occupant	Unbelted occupant	Belted occupant
		$\dot{x}_{ov} = 0$	$\dot{x}_{ov} = 0$
		$\dot{x}_{ov} = 0$	$\dot{x}_{ov} = 0$
		$\dot{x}_v = \int_0^C \ddot{x}_v \cdot dx_v$ $\dot{x}_{ov} = \int_0^C (\ddot{x}_o - \ddot{x}_v) dx_v$	$\dot{x}_v = \int_0^C \ddot{x}_v \cdot dx_v$ $\dot{x}_{ov} = \int_0^C \int_0^d (\ddot{x}_o - \ddot{x}_v) dx_v$

In order to determine the kinematic parameters influencing occupant injury in the event of a frontal impact, the phenomena that occur during the collision are analyzed in terms of the first impact which is vehicle barrier and secondary impact which is occupant-vehicle. Thus, the following hypotheses can be formulated:

1) *The first vehicle-barrier impact:*
    a) *vehicle hits barrier and frontal structure deformation occurs;*
    b) *the vehicle's kinetic energy is consumed by deforming frontal structure;*
2) *Second Impact is occupant - vehicle:*
    a) *when the vehicle hits the barrier is decelerated sharply, and the occupant continues to move due to inertial forces and hits the interior of the vehicle, or its movement is limited by restraint systems;*
    b) *part of the occupant's kinetic energy is consumed by deforming internal parts of the car body or by compression of the occupant body;*
    c) *the remaining kinetic energy is dissipated in the deceleration of the occupant-vehicle system;*
    d) *kinetic energy dissipated during the second impact is directly proportional to the mass and relative velocity of occupant-vehicle.*

Based on these considerations, creating a mathematical model to determine the kinematic parameters needed to estimate the risk of occupant injury in frontal impact conditions, is one of the main goals of this paper.

III. MATHEMATICAL MODELING OF FRONTAL IMPACT OF TWO VEHICULE

Changes in acceleration at the passenger compartment of the vehicle was determined using a simplified mechanical model, consisting of two bodies connected by elastic elements by an elastic spring. Simplifying assumptions of the mechanical model are the following [4]:

1) *the impact between the vehicles was considered to be central-right with total overlap to the frontal surfaces;*
2) *masses of vehicles are considered to be concentrated in the center of mass;*
3) *the force of collision (F_c) acting on the center of mass of the vehicle;*
4) *was considered that the deformation front of the vehicle structure shall be a linear spring model ("Fig 1") [5].*

The equations of motion of the vehicle in the collision are obtained based on generalized Lagrange equation [6], [7]:

$$\frac{d}{dt} \cdot \frac{\partial E_C}{\partial \dot{q}_i} - \frac{\partial E_C}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \tag{1}$$

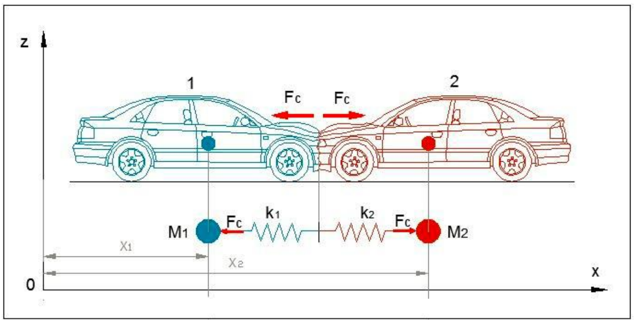


Fig. 1. The scheme of the mathematical model for vehicle-vehicle impact

The significance of quantities involved in the Lagrange equation and in “Fig 1” are:

E_c – the kinetic energy of the system;

V – potential energy of the system;

q_i – generalized coordinates.

Generalized coordinates of the model according to the scheme of collision from “Fig 1” are:

x_1 – moving vehicle 1 Ox direction [m];

x_2 – moving vehicle 2 Ox direction [m];

$$q_i = x_1, x_2 \tag{2}$$

M_1, M_2 – masses of vehicles [kg];

k_1, k_2 – vehicles frontal structural rigidity [N/m].

The kinetic energy or potential energy of the system consisting of the two vehicles are:

$$E_C = \frac{1}{2} \cdot M_1 \cdot \dot{x}_1^2 + \frac{1}{2} \cdot M_2 \cdot \dot{x}_2^2 \tag{3}$$

$$V = \frac{1}{2} \cdot k_1 \cdot \delta_1^2 + \frac{1}{2} \cdot k_2 \cdot \delta_2^2 \tag{4}$$

where: δ_1, δ_2 - the dynamic deformations of vehicles.

The total dynamic deformation during the collision is:

$$\begin{aligned} \delta &= \delta_1 + \delta_2 \\ \delta &= x_1 - x_2 \end{aligned} \tag{5}$$

Considering k – stiffness equivalent to the pair of vehicles in collisions, we obtain:

$$V = \frac{1}{2} \cdot k \cdot (x_1 - x_2)^2 \tag{6}$$

It is believed that upon the vehicles, during the impact, acts only one force of collision, other forces being neglected compared to this.

Lagrange equation for vehicle 1 is:

$$M_1 \cdot \ddot{x}_1 \cdot \dot{x}_1 + k \cdot \dot{x}_1 \cdot (x_1 - x_2) = -M_1 \cdot \ddot{x}_1 \tag{7}$$

Balance resulting from external forces:

$$\begin{aligned} F_{1C} &= -F_{2C} \\ M_1 \cdot \ddot{x}_1 &= -M_2 \cdot \ddot{x}_2 \end{aligned} \tag{8}$$

$$\begin{cases} \ddot{x}_1 = -\dfrac{k \cdot \dot{x}_1 (x_1 - x_2)}{M_1 \cdot (\dot{x}_1 + 1)} \\ \ddot{x}_2 = \dfrac{k \cdot \dot{x}_1 (x_1 - x_2)}{M_2 \cdot (\dot{x}_1 + 1)} \end{cases} \tag{9}$$

Thus obtain a system of two differential equations of second degree where the unknowns are the accelerations, velocities and movements of the two vehicles (9).

To solve this system were made the following substitutions:

$$x_1 = x(1), \dot{x}_1 = x(2), x_2 = x(3), \dot{x}_2 = x(4) \tag{10}$$

The two differential equations of the second degree being transformed into a system of four differential equations of first degree as:

$$\begin{cases} dy(1) = x(2) \\ dy(2) = -\dfrac{k \cdot x(2) \cdot (x(1) - x(3))}{M_1 \cdot (x(2) + 1)} \\ dy(3) = x(4) \\ dy(4) = \dfrac{k \cdot x(2) \cdot (x(1) - x(3))}{M_2 \cdot (x(2) + 1)} \end{cases} \tag{11}$$

In this form, the system can be solved by numerical integration using the Runge-Kutta method.

IV. Mathematical Modeling of the Impact of Occupant-Vehicle

Study carried out by researchers at Autoliv Sweden [8], driver protection in frontal impact of relative impact speeds above 80 km / h shows that the main parameters of the inner passive safety systems that affect acceleration occupant head and chest during impact is maximum reaction force of the steering column and the maximum forces and diagonal branches of the hip belt. The values of these parameters and their influence on occupant acceleration are shown in Table II.

For mathematical modelling of the impact of occupant-vehicle was considered occupant simiplified model, shown in “Fig 2”, as consisting of two rigid bodies.

The lower limbs and pelvis are represented by body mass m_3, and trunk, head, neck and upper limbs are represented by body mass m_4. It was considered that the masses of the two bodies are concentrated in the centers of mass.

TABLE II
THE KINEMATIC PARAMETERS OF THE OCCUPANT AND THE PARAMETERS OF INFLUENCE

Kinematic parameters of the occupant	The parameters of influence	Influence	Limit values
Head acceleration a_C [g]	F_v - reaction force of the steering wheel	direct proportional	5...8 kN
	F_s - limit force of the seatbelt on the hip	inverse proportional	3...6 kN
	F_u - shoulder belt force limit	inverse proportional	5...8 kN
Thorax acceleration a_T [g]	F_s - limit force of the seatbelt on the hip	direct proportional	3...6 kN
	F_u - shoulder belt force limit	direct proportional	5...8 kN

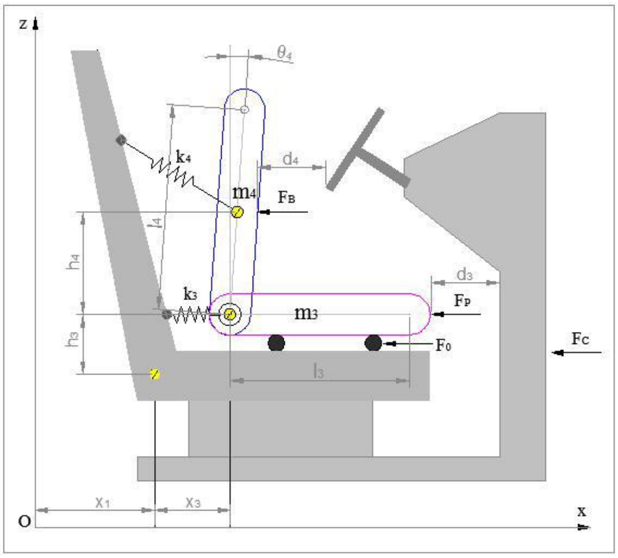


Fig. 2. The scheme of the mathematical model occupant-vehicle

The meaning of the notations used in "Fig 2" is:

x_1 – moving vehicle [m];

x_3 – pelvic movement of the occupant (including pelvis and lower limbs) [m];

θ_4 – relative angular movement of the trunk (head, torso and upper limbs) to the pelvis [rad];

k_3 – branch stiffness hip belt [N/m];

k_4 – branch stiffness shoulder belt [N/m];

F_0 – friction between the occupant and the seat [N];

F_B – the reaction force of arms mounted on the steering wheel [N];

F_P – car floor reaction force [N];

F_C – the force applied to the vehicle collision [N];

d_4 – distance between the occupant's chest and the steering wheel [m];

d_3 – distance between the occupant's knees and knee protection [m];

l_3 – length of segment 3 [m];

l_4 – length of segment 4 [m];

h_3 – height of the center of mass of the body 3 [m];

h_4 – height of the center of mass of the body 4 [m];

Was considered that upon occupant during impact, the following forces act:

1) *vehicle floor reaction force transmitted through the legs (F_P);*
2) *steering reaction force transmitted through the arms (F_B);*
3) *force of friction between the lower body and the seat occupant (F_0);*
4) *reaction force of the belt, which was considered by industry rigidities shoulder (k_4) and the branch of the hip (k_3).*

The equations of motion of mass m_3 and m_4 bodies will be written in the integrated mathematical model.

V. INTEGRATED MATHEMATICAL MODEL FOR THE STUDY OF FRONTAL IMPACT

To align the vehicle occupants kinematics is useful analytical description of the entire system consisting of both vehicles and their occupants.

For the determination of kinematic parameters and dynamic occupant in the vehicle frontal impact with another vehicle, vehicle-to-vehicle impact model and model-vehicle occupant impact, above, were merged into a unified mathematical model which basically outline is presented in "Fig 3".

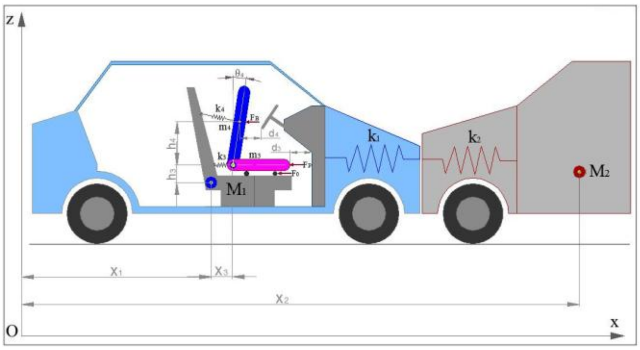


Fig. 3. The scheme of the integrated mathematical model

Significance notations used is the same as in the previous section of the paper.

The equations of motion of the vehicle in the collision are obtained based on generalized Lagrange equation (1), considering as generalized coordinates:

x_1 - moving vehicle 1 *Ox* direction;

x_2 - moving vehicle 2 *Ox* direction;

x_3 - relative movement in the direction *Ox* body mass m_3 of the occupant relative to the vehicle;

θ_4 - relative angular movement of the body 3 relative to body 4.

The kinetic energy of the system is:

$$E_C = \frac{1}{2}\cdot M_1 \cdot \dot{x}_1^2 + \frac{1}{2}\cdot M_2 \cdot \dot{x}_2^2 + \frac{1}{2}\cdot m_3 \cdot (\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}\cdot m_4 \cdot \left(\dot{x}_1 + \dot{x}_2 + \frac{l_4}{2}\cdot \dot{\theta}_4 \cdot cos\theta_4 \right)^2 \tag{12}$$

The potential energy of the system is:

$$V = \frac{1}{2}\cdot k \cdot (x_1 - x_2)^2 + \frac{1}{2}\cdot k_3 \cdot \dot{x}_3^2 + \frac{1}{2}\cdot k_4 \cdot \left(x_3 + \frac{l_4}{2}\cdot \dot{\theta}_4 \cdot cos\theta_4 \right)^2 + m_3 \cdot g \cdot h_3 + m_4 \cdot g \cdot \left(h_3 + \frac{l_4}{2}\cdot cos\theta_4 \right) \tag{13}$$

The forces acting on the rigid bodies are collision force (F_C), the friction force between the occupant and the seat (F_0), vehicle floor reaction force (F_P), a steering reaction force (F_B), the reaction force of the joint of the occupant (R) and the forces of the belt (F_{K3}, F_{K4}).

$$F_C = -M_1 \cdot \ddot{x}_1 = M_2 \cdot \ddot{x}_2 \tag{14}$$

Such generalized Lagrange equation (1) for the four objects considered is:

$$\begin{cases} \ddot{x}_1 = -\dfrac{k\cdot \dot{x}_1 \cdot (x_1 - x_2) - (F_0 + F_P + F_B)}{M_1 \cdot (x_1 + 1) + m_3 \cdot (\dot{x}_1 + \dot{x}_3) + m_4 \cdot \left(\dot{x}_1 + \dot{x}_3 + \frac{1}{2}\cdot l_4 \cdot \dot{\theta}_4 \cdot cos^2\theta_4 \right)} \\ \ddot{x}_2 = \dfrac{k\cdot \dot{x}_1 \cdot (x_1 - x_2) - (F_0 + F_P + F_B)}{M_2 \cdot (x_1 + 1) + m_3 \cdot (\dot{x}_1 + \dot{x}_3) + m_4 \cdot \left(\dot{x}_1 + \dot{x}_3 + \frac{1}{2}\cdot l_4 \cdot \dot{\theta}_4 \cdot cos^2\theta_4 \right)} \\ \ddot{x}_3 = -\dfrac{(k_3 - k_4)\dot{x}_3 \cdot x_3 + k_4 \cdot \frac{l_4}{2}\cdot \dot{x}_3 \cdot sin\theta_4 + m_4 \cdot g \cdot \frac{sin\theta_4}{cos\theta_4} + (F_0 + F_P)}{m_3 \cdot (\dot{x}_1 + \dot{x}_3) + m_4 \cdot (\dot{x}_1 + \dot{x}_3 + \frac{1}{2}\cdot l_4 \cdot \dot{\theta}_4 \cdot cos\theta_4)} \\ \ddot{\theta}_4 = \dfrac{m_4 \cdot \frac{l_4}{2}\cdot \dot{\theta}_4 \cdot sin\theta_4 \cdot (\dot{x}_1 + \dot{x}_3) + m_4 \cdot \frac{l_4^2}{4}\cdot \dot{\theta}_4^2 \cdot sin\theta_4 \cdot cos\theta_4 - k_4 \cdot \frac{l_4}{2}\cdot x_3 \cdot cos\theta_4}{m_4 \cdot \frac{l_4}{2}\cdot cos\theta_4 \left(\dot{x}_1 + \dot{x}_3 + \frac{1}{2}\cdot l_4 \cdot \dot{\theta}_4 \cdot cos\theta_4 \right) + m_4 \cdot \frac{1}{12}\cdot l_4^2 \cdot \theta_4} + \\ + \dfrac{-k_4 \cdot \frac{l_4^2}{4}\cdot sin\theta_4 \cdot cos\theta_4 + m_4 \cdot g \cdot \frac{l_4}{2}\cdot sin\theta_4 - F_B - m_4 \cdot g \cdot \frac{sin\theta_4}{cos\theta_4}}{m_4 \cdot \frac{l_4}{2}\cdot cos\theta_4 \left(\dot{x}_1 + \dot{x}_3 + \frac{1}{2}\cdot l_4 \cdot \dot{\theta}_4 \cdot cos\theta_4 \right) + m_4 \cdot \frac{1}{12}\cdot l_4^2 \cdot \theta_4} \end{cases}$$

(15)

To solve this system have made the following substitutions:

$$\begin{gathered} x_1 = x(1), \dot{x}_1 = x(2), x_2 = x(3), \dot{x}_2 = x(4) \\ x_3 = x(5), \dot{x}_3 = x(6), \theta_4 = x(7), \dot{\theta}_4 = x(8) \end{gathered} \tag{16}$$

Obtained a system of eight differential equations of the first degree, which can be solved by numerical integration using the Runge-Kutta method.

VI. INPUT DATA AND RESULTS OF THE MATHEMATICAL MODEL

System constants and initial parameters of the system are input into the mathematical model. These are summarized in Table III and IV.

TABLE III
SYSTEM CONSTANTS

System constants	Symbol	Unit of measure
masses of the vehicles	M_1, M_2	kg
structural rigidity of the vehicle front	k_1, k_2	N/m
rigidities of the branches of the seatbelt	k_3, k_4	N/m
the forces acting on the occupant	F_0, F_P, F_B	N

TABLE IV
INITIAL PARAMETERS OF THE SYSTEM

Initial parameters of the system	Symbol	Unit of measure
movement of the vehicles	x_{10}, x_{20}	m
vehicles velocities	$\dot{x}_{10}, \dot{x}_{20}$	m/s
linear movement of the pelvis relative to the vehicle 1	x_{30}	m
relative angular movement of the chest to the pelvis	θ_{40}	rad
the linear velocity of the vehicle relative to the pelvis 1	$\dot{x}_{30}$	m/s
relative angular velocity of the thorax to the pelvis	$\dot{\theta}_{40}$	rad/s

The results of the mathematical model or system solutions of equations (15) are shown in Table V.

TABLE V
RESULTS OF THE MATHEMATICAL MODEL

Results of the mathematical model	Symbol	Unit of measure
movement of the vehicles	$x_1, x_2 = f(t)$	m
vehicles velocities	$\dot{x}_1, \dot{x}_2 = f(t)$	m/s
linear movement of the pelvis relative to the vehicle 1	$x_3 = f(t)$	m
relative angular movement of the chest to the pelvis	$\theta_{40} = f(t)$	rad
the linear velocity of the vehicle 1 relative to the pelvis	$\dot{x}_3 = f(t)$	m/s
relative angular velocity of the thorax to the pelvis	$\dot{\theta}_4 = f(t)$	rad/s

Based on these movements and velocities can cause bodies 3 and 4 in the inertial reference system.

By numerical integration of velocities are obtained accelerations of the four bodies of the system.

Parameters of the vehicle and occupant kinematics can be used to assess the risk of occupant injury, depending on the initial impact.

VII. CONCLUSION

Kinematic parameters of the vehicle are a direct result of the impact between the two vehicles according to their masses, the relative speed of impact and stiffness coefficients.

Occupant kinematic parameters are determined by kinematic parameters of the vehicle seat belt stiffness, mass and height of the occupant.

The mathematical model developed has the following limits:

1) *is a two-dimensional model, multi-body with concentrated masses;*
2) *occupant's head and torso are treated with the same rigid body;*
3) *movement of the body 3 - pelvis horizontal axis Ox, is considered without taking into account the angle of the seat;*
4) *belt forces and reaction forces are considered to act on the application points Ox and have centers of mass of bodies 3 and 4.*
5) *reaction forces acting on the occupant of the vehicle are considered constant;*
6) *belt rigidity expressed by coefficients k_3 and k_4, is considered constant.*

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